Exactly solvable maps of on-off intermittency

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A solvable three-dimensional deterministic map that shows on-off intermittency is derived from a coupled mapping system. The distribution of an on-off variable, that of laminar duration, and the mean laminar duration are analytically obtained. The map enables us to discuss the probability measure on the attractor, which also gives the singularity spectrum $f(\alpha)$. The singularity of the measure caused by the on-off intermittency give rises to a linear part of the spectrum $f(\alpha)$. [S1063-651X(97)10005-8]

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Recently, on-off intermittency was observed in various nonlinear dynamical systems [1–9]. This intermittency was first observed numerically [1] and experimentally [7] in a coupled system consisting of chaotic oscillators, when the synchronized state is broken. When the synchronized state is stable, the chaotic attractor is located in a smooth invariant manifold of lower dimension than that of the full phase space [5]. Slightly beyond the critical point, at which the synchronized state becomes unstable for the perturbation out of the invariant manifold, the orbit escapes far away from the invariant manifold (burst or *on* state) but returns to its neighborhood and stays there for a long time (laminar phase or *off* state). This temporal evolution is repeated in an irregular manner and called on-off intermittency.

Such behavior is characterized by the distribution of an on-off variable (the distance from the invariant manifold) $P(x)$ and that of laminar duration $P_\tau(\tau)$, which is studied theoretically by use of stochastic or random models [1,4]. We propose here a three-dimensional deterministic map, which is derived from a coupled map. The map enables us to obtain the exact expressions of $P(x)$ and $P_\tau(\tau)$, which agree with the results from the stochastic models. Further, we can discuss the probability density on the attractor and the structure of the phase space; and get the singularity spectrum $f(\alpha)$, which has a singular linear part.

The starting coupled mapping system is the Poincaré map of coupled chaotic oscillators,

$$U_1(t+1)=F(U_1(t))+C\{F(U_2(t))-F(U_1(t))\}, \quad (1a)$$

$$U_2(t+1)=F(U_2(t))+C\{F(U_1(t))-F(U_2(t))\}, \quad (1b)$$

where $U_j(t)$ is the state vector of the jth chaotic oscillator at the discrete time t, $F(U)$ is a mapping function, and C is a coupling matrix. This map (1) always has the synchronized solution $U_1(t)=U_2(t)$. Introducing new vectors $U\equiv(U_1+U_2)/2$ and $V\equiv(U_1-U_2)/2$, and linearizing for V around the synchronized solution $V(t)=0$, we have

$$U(t+1)=F(U(t)), \quad (2a)$$

$$V(t+1)=(1-2C)DF(U(t))V(t), \quad (2b)$$

where $DF(U(t))$ is the Jacobian matrix of F at $U(t)$. The synchronized solution lies in the invariant manifold $V=0$ and the stability in it is indicated by the largest Liapunov exponent Λ_1 of Eq. (2a). The deviation $|V(t)|$ from the invariant manifold is exponentially expanded or contracted as $\exp[\lambda_\perp t]$. We call this stability exponent $\lambda_\perp$ the transverse Liapunov exponent. When the synchronization is unstable, i.e., $\lambda_\perp>0$, the orbit is repelled from the invariant manifold on average. However, if $\lambda_\perp$ is a small positive and the global structure of the system has the chaotic reinjection process into the neighborhood of the invariant manifold, on-off intermittency appears. That is why the model of on-off intermittency needs the form (2) around the invariant manifold and the suitable reinjection process.

Here, we take $U=(X,Y), V=(x,y)$ and $F(U)$ is a baker's type map,

$$X(t+1)=G(X(t))=\begin{cases}\dfrac{X(t)}{a} & [0\leqslant X(t)<a],\\[2ex] \dfrac{1-X(t)}{1-a} & [a\leqslant X(t)<1],\end{cases} \quad (3a)$$

$$\begin{aligned}Y(t+1)&=H(Y(t),X(t))\\ &=\begin{cases}JaY(t) & [0\leqslant X(t)<a],\\ \frac{1}{2}+J(1-a)Y(t) & [a\leqslant X(t)<1],\end{cases}\end{aligned} \quad (3b)$$

where $0<a<\frac{1}{2}, 0<J<1$. When C is a scalar c, Eq. (2b) is reduced to $[x(t+1), y(t+1)] = c'\{[dG(X(t))/dX]x(t), [\partial H(Y(t), X(t))/\partial Y]y(t)\}$, where $c' = 1-2c$. The variable y vanishes, because $y(t)\to 0$ as $t\to\infty$. Since the sign of x is not important, we replace $|x|$ by x. We append a simple reinjection process to this dynamics, and choose suitable values to a and c', as the dynamics of x has the Markov condition that is mentioned later. Finally, we get

$$x(t+1) = g(x(t), X(t)) = \begin{cases} b^{-2}x & (0\le x<b^2, 0\le X<a), \\ bx & (0\le x<b^2, a\le X<1), \\ \dfrac{b-x}{b(1-b)} & (b^2\le x<b), \\ \dfrac{x-b}{1-b} & (b\le x<1), \end{cases} \tag{3c}$$

where $b=[a/(1-a)]^{1/3}, c' = a^{1/3}(1-a)^{2/3}$. For $a<a_c=\frac{1}{3}$, the synchronized solution $x=0$ is stable to the perturbation along the x direction, because its transverse Liapunov exponent is given by $\lambda_\perp \equiv \langle\lambda(t)\rangle = \Lambda_1 - \ln c' = (1-3a)\ln b$, where $\langle\ \rangle$ denotes the long time average $\langle h(t)\rangle \equiv \lim_{T\to\infty}(1/T)\Sigma_{t=0}^{T-1}h(t)$, $\lambda(t)\equiv \ln|\partial g(x(t),X(t))/\partial x|$. And the fluctuations of the transverse Liapunov exponent are characterized by its variance $D\equiv\langle[\Sigma_{t=0}^{T-1}\lambda(t)-\lambda_\perp T]^2\rangle/2T = \frac{1}{2}\langle[\lambda(t)-\lambda_\perp]^2\rangle = \frac{9}{2}a(1-a)(\ln b)^2$. For $a>a_c$, the orbit is repelled from $x=0$ and returns to $x\simeq 0$ by the reinjection process ($b^2\le x<1$), thus the on-off intermittency appears, whose time series $x(t)$ and attractor are shown in Fig. 1.

First, we discuss the probability density $\rho(X,Y,x)$ on the attractor. As the dynamics [Eqs. (3a), (3b)] is independent of x and similar to the famous generalized baker's transformation, $\rho(X,Y,x)$ is uniform along the X direction so that $\rho(X,Y,x)=\rho(Y,x)$ and has the fractal structure along the Y direction. The dynamics (3c) is independent of Y, thus we consider only X and x in the discussion of on-off intermittency. We define $P(x)\equiv\int dY\rho(Y,x)$ and introduce the regions $R_{1,j}=[0\le X<a, x_{j+1}\le x<x_j]$, $R_{2,j}=[a\le X<1, x_{j+1}\le x<x_j]$, $R_j=R_{1,j}+R_{2,j}$, $R=[0\le X<1, 0\le x<1] = \Sigma_{j=0}^{\infty}R_j$, where $x_j\equiv b^j, (j=0,1,2,\dots)$. The regions $R_{1,j}, R_{2,j}, (j=2,3,4,\dots)$ are mapped onto R_{j-2}, R_{j+1}, respectively, and $R_j (j=0,1)$ onto R, so that the regions have the Markov condition. For the region R_j, we define the characteristic function $E_j(x)$, which is equal to 1 if $x\in R_j$, and 0 otherwise. Then the dynamics (3) has the probability density in the form $P(x)=\Sigma_{j=0}^{\infty}p_jE_j(x)$ and p_j satisfy the equations

$$p_j = \begin{cases} \ell_0 p_0 + \ell_1 p_1 + ab^2 p_{j+2} + (1-a)b^{-1}p_{j-1} & (j=3,4,5,\dots) \\ \ell_0 p_0 + \ell_1 p_1 + ab^2 p_{j+2} & (j=0,1,2) \end{cases} \tag{4}$$

where $\ell_j = x_j - x_{j+1} = (1-b)b^j$. The solution of Eq. (4) is given by

$$p_j = \begin{cases} q + r\nu^{j-2} & (j=2,3,4,\dots) \\ q + r_j & (j=0,1), \end{cases} \tag{5}$$

$$\nu = \frac{\sqrt{(4-3a)a}-a}{2ab}, \quad q = -b\frac{1-ab^2\nu^2}{1-a}r,$$

where $r_0=\{1+ab^2(1-\nu^2)\}r, r_1=\{1+ab^2\nu(1-\nu)\}r$. [The value of r is given by the normalized condition $1=\int_0^1 dxP(x)=\Sigma_{j=0}^{\infty}p_j\ell_j$.] The probability density $P(x)$ is the distribution of the on-off variable x and shows a scaling form $P(x)\sim x^\eta, \eta=(\ln|\nu|/\ln b)$ for $x\ll 1$. Near the critical point $a=a_c$, the exponent η has the form $\eta=-1+(3/\ln 2)\,\epsilon$ for $\epsilon\equiv(a-a_C)/a_C\to 0$, which agrees with the result from the stochastic model $\eta=-1+\lambda_\perp/D$ [1], since $\lambda_\perp\to(\ln 2/3)\epsilon$ and $D\to(\ln 2/3)^2-(\ln 2/3)\epsilon$ for $\epsilon\to 0$.

Next, we discuss the laminar duration. We assign the laminar region to $x<x_2$, and τ_j denotes the mean laminar duration of the orbits starting from R_j. The dynamics (3) leads the equations

$$\tau_j = a\tau_{j-2} + (1-a)\tau_{j+1} + 1 \quad (j=2,3,4,\dots). \tag{6}$$

We solve the difference equation (6) with the boundary condition $\tau_0=\tau_1=\tau_n=0$. Taking the limit $n\to\infty$, we get $\tau_j=[j-(1-\phi^j)/(1-\phi)]/(3a-1), \phi=[a-\sqrt{(4-3a)a}]/2(1-a) = -ab\nu/(1-a)$, where ϕ is one of the solutions of the characteristic equation of Eq. (6). As the reinjection is uniform for $0<x<b^2$, the probability of finding the initial point of a laminar motion in R_j is proportional to ℓ_j. Therefore, the mean value of the laminar duration is given by

$$\overline{\tau} = \sum_{j=2}^{\infty}\tau_j\ell_j b^{-2} = \frac{1-\phi}{(1-b\phi)(3a-1)}. \tag{7}$$

Since $\phi\to-\frac{1}{2}, b\to(\frac{1}{2})^{1/3}$ for $\epsilon\to 0$, the mean laminar duration diverges as $\overline{\tau}\propto 1/(3a-1)=1/\epsilon, (\epsilon\to 0)$, which also agrees with the result of the stochastic model [4].

We can also discuss the distribution of the laminar duration $P_\tau(\tau)$. Let $p_j(\tau)$ be the probability that the laminar motion lasts for τ with the initial point contained in R_j. Then Eq. (3) leads to the equation

$$p_j(\tau+1) = ap_{j-2}(\tau) + (1-a)p_{j+1}(\tau) \quad (j=2,3,4,\dots), \tag{8}$$

where we put $p_1(\tau)=p_0(\tau)=\delta_{\tau,0}$. Summing up the both sides of Eq. (8) multiplied by $s^{\tau+1}$ with respect to τ, and introducing the generating function $Q_j(s)\equiv\Sigma_{\tau=0}^{\infty}p_j(\tau)s^\tau$, we have the difference equation

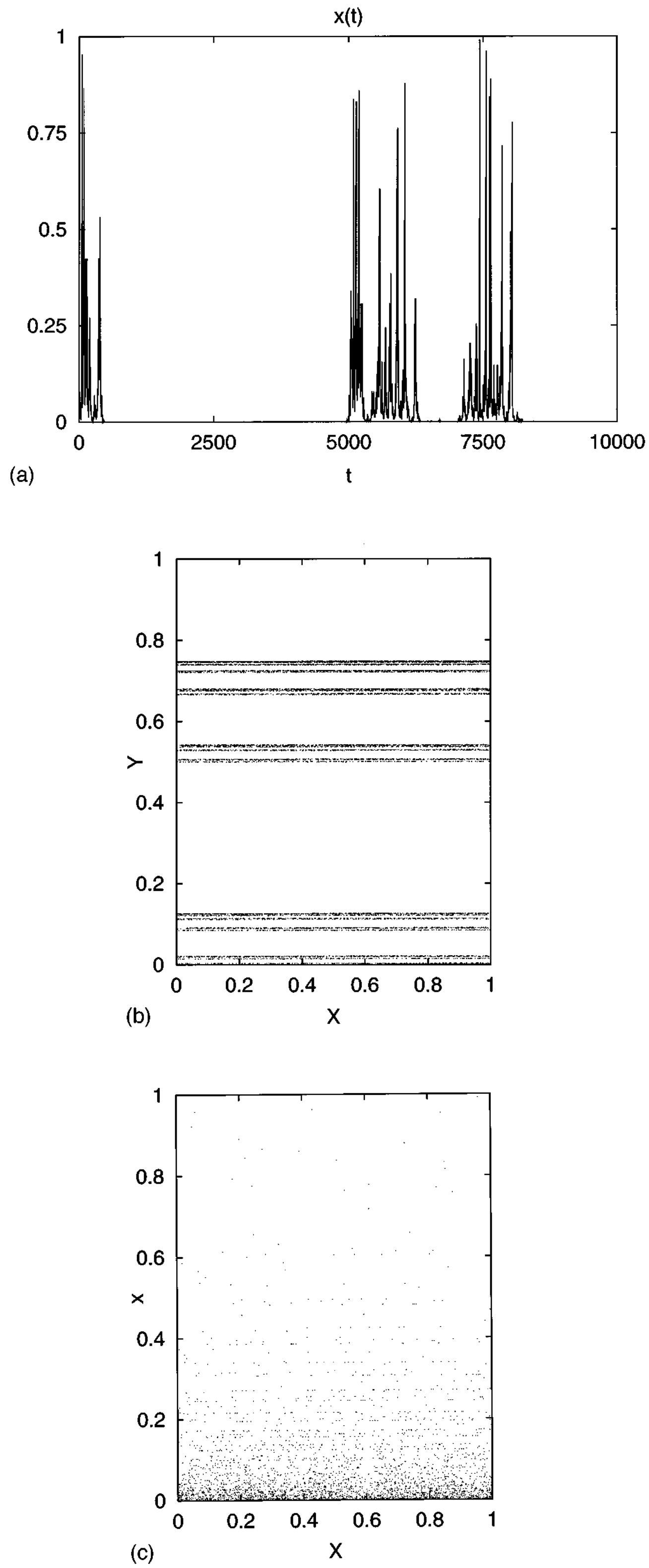

FIG. 1. (a) Time series $x(t)$ of the map (3) with $\epsilon=(a-a_c)/a_c=0.01$. (b), (c) Projections of the attractor with $\epsilon=0.01, J=0.5$, which has 10^4 points. It has the fractal structure along the Y direction and its measure has a singularity along the x direction.

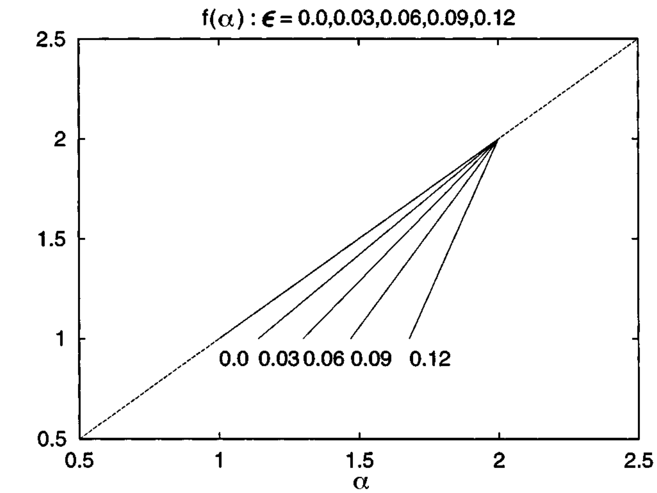

FIG. 2. The singularity spectrum $f(\alpha)$ for the attractor of the map (3) in the strong dissipation limit $J\rightarrow 0$ with $\epsilon=0.0,0.03,0.06,0.09,0.12$. The $f(\alpha)$ has a liner part with the slope $s=-1/\eta$ for $a<a_0$, which is tangent to $f=\alpha$ at $a=a_c=1/3$.

$$Q_j(s)=asQ_{j-2}(s)+(1-a)sQ_{j+1}(s) \quad (j=2,3,4,\ldots), \tag{9}$$

with the boundary conditions $Q_0(s)=Q_1(s)=1$. The characteristic equation of (9) $1=as\xi^{-2}+(1-a)s\xi$ has three real solutions, $\xi_1(s)>1>\xi_2(s)>0>\xi_3(s)>-1$ for $0<s<1$. Therefore, the solution of Eq. (9) is $Q_j(s)=\Sigma_{k=1,2,3}A_k\xi_k(s)^j$. Since $Q_j(s)$ is finite for $0<s<1$, A_1 must be equal to 0. The relations $A_2=(1-\xi_3)/(\xi_2-\xi_3)$, $A_3=-(1-\xi_2)/(\xi_2-\xi_3)$ are obtained from the boundary conditions. Introducing the generating function of $P_\tau(\tau)$ as $Q(s)\equiv\Sigma_{\tau=0}^{\infty}P_\tau(\tau)s^\tau$, we have $Q(s)=\Sigma_{j=2}^{\infty}(\ell_j/b^2)Q_j(s)=(1-b)\Sigma_{k=2,3}A_k\xi_k^2/(1-b\xi_k)$. In the vicinity of the critical point, the laminar duration becomes very long. Thus we are interested in the asymptotic behavior of $P_\tau(\tau)$ for $\tau\gg 1$. For this purpose we expand $Q(s)$ around Δs with $s=1-\Delta s(\Delta s\ll 1)$, so that we have $Q(s)\simeq\{1-(4-b)\sqrt{\Delta s}/3(1-b)\}$, since $\xi_2\simeq 1-\sqrt{\Delta s}, \xi_3\simeq -1/2+\Delta s/6$. Using the relation $\sqrt{1-s}=\Sigma_{n=1}^{\infty}2^{-2n+1}(2n-2)!s^n/n!(n-1)!$ and Stirling's formula, we obtain the final result

$$P_\tau(\tau)\propto\tau^{-3/2}. \tag{10}$$

This result also agrees with the stochastic, numerical [4] and experimental results [8,9].

Last, we discuss the singularity spectrum $f(\alpha)$ [10]. The singular local structures of chaotic attractors have been found to produce remarkable linear parts in $f(\alpha)$ [11,12]. The attractor of on-off intermittency also has the singular local structure on the synchronized solution $x=0$, which suggests $f(\alpha)$ has a linear part. In this paper, we only discuss a simple case of the attractor for the strong dissipation limit $J\rightarrow 0$. Then, the attractor lies in the subspace $Y=0$ and $Y=1/2$ and the probability density on the attractor is given by $\rho(X,Y,x)=\{\delta(Y)+(1-a)\delta(Y-1/2)\}\Sigma_{j=0}^{\infty}p_jE_j(x)$ and Eq. (5). Let us divide the subspace into the squares S_m with measure M_m and sides $\kappa=b^n=x_n\ll 1$. The measure M_m on all of these squares except for that adjoining $x=0$ is approximated by $\rho(X,Y,x)\kappa^2$. Then, the partition function for the measure is given by

$$\chi(q,\kappa)\equiv\sum_m M_m^q$$
$$\simeq\{a^q+(1-a)^q\}\left[\frac{1}{\kappa}\left(\kappa\sum_{j=n}^{\infty} p_j\ell_j\right)^q+\sum_{j=0}^{n-1}\frac{\ell_j}{\kappa^2}(p_j\kappa^2)^q\right]$$
$$\sim\kappa^{(2+\eta)q-1}+\kappa^{2(q-1)}. \qquad (11)$$

In the limit $\kappa\to 0$, since the partition function $\chi(q,\kappa)$ has a scaling form $\chi(q,\kappa)\sim\kappa^{\zeta(q)}, \zeta(q)=(q-1)D_q=\min\{\alpha_* q-1, 2(q-1)\}$, where $\alpha_*=2+\eta$ and D_q is the generalized dimension. The Legendre transformation $\alpha(q)=d\zeta(q)/dq$ and $f(\alpha)=\alpha q-\zeta(q)$ leads to the spectrum $f(\alpha)$

$$f(\alpha)=1+\frac{\alpha-\alpha_*}{2-\alpha_*} \quad (\alpha_*\leqslant\alpha\leqslant 2). \qquad (12)$$

It has a remarkable linear slope $s=-1/\eta$, which adjoins the singular exponent $\alpha=\alpha_*$ at the synchronized state $x=0$ to $\alpha=2$ at $x>0$. Therefore, when the parameter a approaches from $a>a_c$ to the critical point $a=a_c$, the probability density $\rho(X,x)$ has a singularity on the $x=0$ for $a<a_0$ where $a=a_0$ is given by $\nu=1$ and the linear part of $f(\alpha)$ appears for $a<a_0$. At the critical point $a=a_c$, the linear part is tangent to $f=\alpha$ and disappears for $a<a_c$. [For $a<a_c$, $f(\alpha)$ is reduced to a single point $f=\alpha=1$.] These are shown in Fig. 2.

We discussed some exact results of on-off intermittency for the deterministic dynamical system (3). Even replacing the map (3a) by another solvable piecewise-linear map [13], we expect to obtain similar results. Imposing the boundary condition $p_0(\tau)=p_1(\tau)=p_n(\tau)=\delta_{\tau,0}$ on Eq. (8), we can discuss the escape rate from the region $x_n\leqslant x<1$, which characterizes the transient motion to the stable synchronized state $x=0$ for $a<a_c$. For the two dimensional map, the spectrum $f(\alpha)$ closely relates to the spectrum $\psi(\lambda)$ of the local expansion rates λ [12]. We can discuss the similar relation for the on-off intermittency. The linear part of $f(\alpha)$ shows similar information to $P(x)$. However, if the smooth invariant manifold is curved in the full phase space, it is difficult to observe the on-off variable and we cannot get $P(x)$ and $P_\tau(\tau)$. Then, it is more important to discuss the linear part of $f(\alpha)$. It is very interesting to explore through other examples whether our results are indeed germane to the case where the invariant manifold is not flat. We will consider this point in the next occasion. Since we can construct the Frobenius-Perron operator of the map (3a), (3c), we expect to obtain the analytical results for the power spectrum of $x(t)$ and we can also apply the thermodynamical formalism [14,15] to the present model analytically. These are planned to be reported elsewhere.

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